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On semiprimary rings of finite global dimension

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Abstract

Suppose that A is a semiprimary ring satisfying one of the two conditions: 1) its Yoneda ring is generated in finite degrees; 2) its Loewy length is less or equal than three. We prove that the global dimension of A is finite if, and only if, there is a $m > 0$ such that $Ext_A^n(S, S) = 0$, for all simple A-modules S and all $n \geq m$

In a recent paper, Skowronski, Smal∅ and Zacharia ([8]) proved that a left Artinian ring A has finite global dimension if, and only if, every finitely generated indecomposable left A-module has either finite projective dimension or finite injective dimension. Then, they showed that one cannot replace 'indecomposable' by 'simple' in that statement, by giving a counterexample of Loewy length 4. Finally they asked the following question:

Question: Suppose A is a left Artinian ring such that, for each finitely generated indecomposable left A-module M, one has $Ext_A^n(M, M) = 0$ for $n \gg 0$ (i.e. there is a $m = m(M) > 0$ such that $Ext_A^n(M, M) = 0$ for all $n \geq m$). Does A have finite global dimension?

The two main results of this paper, Theorems 1 and 2, give two partial affirmative answers to the latter question, even in the more general context of semiprimary rings, showing in the way that the above mentioned counterexample is of minimal Loewy length.

All rings in the paper are associative with unit. A ring A is **semiprimary** when its Jacobson radical $J = J(A)$ is nilpotent and A/J is semisimple. In that case, the minimal n such that $J^n = 0$ is called the **Loewy length** of A. Recall that a semiprimary ring is a particular instance of (left and right) perfect ring and, hence, every A-module M has a projective cover $\epsilon : P_0 = P(M) \longrightarrow M$ and a minimal projective resolution $...P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} ...P_1 \xrightarrow{d_1}$

$P_0 \xrightarrow{\epsilon} M \to 0$, both uniquely determined up to isomorphism. We shall put $\Omega^n M = Im(d_n) = Ker(d_{n-1})$, for all $n > 0$, and $\Omega^0 M = M$. By dimension shifting, we have $Ext_A^n(M, X) \cong Ext_A^1(\Omega^{n-1}M, X)$, for every A-module X and every $n > 0$. When X is semisimple and we apply the functor $Hom_A(-, X)$ to the canonical exact sequence $0 \to \Omega^n M \xrightarrow{j} P_{n-1} \longrightarrow \Omega^{n-1}M \to 0$, the induced map $j^* : Hom_A(P_{n-1}, X) \longrightarrow Hom_A(\Omega^n M, X)$ is the zero map, so that $Ext_A^1(\Omega^{n-1}M, X) \cong Hom_A(\Omega^n M, X)$, for all $n > 0$. From that we get that $Ext_A^n(M, X) \cong Hom_A(\Omega^n M, X)$, for all $n \geq 0$. All throughout the paper, whenever necessary, we shall see the latter isomorphism as an identification. On the other hand, the perfectness of A implies that flat and projective modules coincide, from which it follows that the left and right global dimensions coincide with the weak global dimension, which is left-right symmetric (cf [7][Theorem 9.15]). We shall deal only with left modules, hitherto just called 'modules', although the results obtained are left-right symmetric. Since the radical filtration on any module is finite and has semisimple factors, the global dimension of A is the supremum of the projective dimensions of the simple A-modules. As a consequence, that global dimension is finite if, and only if, $Ext_A^n(A/J, A/J) = 0$ for some $n > 0$.

Recall that, for A-modules M, N, P and $m, n \geq 0$, one has a $\mathbf{Z}$-bilinear **Yoneda product** $Ext_A^m(N, P) \times Ext_A^n(M, N) \longrightarrow Ext_A^{m+n}(M, P)$ (cf. [6][Chapter III, Section 5]). We shall need an explicit description of this product when N, P are semisimple, using the above mentioned identification. In this case, if $f \in Hom_A(\Omega^n M, N) \cong Ext_A^n(M, N)$, then the comparison theorem (cf. [6][Theorem III.6.1]) yields a chain map between the minimal projective resolutions of $\Omega^n M$ and M, which is uniquely determined up to homotopy. Then we get a morphism $\tilde{f} : \Omega^{m+n}M \longrightarrow \Omega^m N$. If now $g \in Hom_A(\Omega^m N, P) \cong Ext_A^m(N, P)$ then the Yoneda product $g \cdot f$ is just the composition $g \circ \tilde{f}$ and does not depend on the choices made to get $\tilde{f}$. It is well-known that the Yoneda product makes $E = Ext_A^*(A/J, A/J) = \oplus_{n \geq 0} Ext_A^n(A/J, A/J)$ into a graded ring, with the obvious grading. It is called the **Yoneda ring** of A and $Ext_A^*(M, A/J) = \oplus_{n \geq 0} Ext_A^n(M, A/J)$ is canonically a graded (left) E-module, for every A-module M. In case A is an algebra over a commutative ring K, the grading of E makes it into a graded K-algebra.

Definition 1. *A positively graded ring $R = \oplus_{n \geq 0} R_n$ will be said to be **generated in finite degrees** when there is a $m \geq 0$ such that the subgroup $R_0 \oplus ... \oplus R_m$ generates R as a ring, i.e., there is no proper subring of R containing $R_0 \oplus ... \oplus R_m$*

Remark 1. *If K is a commutative ring and $R = \oplus_{n \geq 0} R_n$ is a positively graded K algebra such that each R_n is a finitely generated K-module, then R is generated in finite degrees if, and only if, R is finitely generated as a K-algebra*

The following is the first main result of the paper:

Theorem 1. *Let A be a semiprimary ring. The following assertions are equivalent:*

1. The global dimension of A is finite

2. The Yoneda ring $E = Ext^*(A/J, A/J)$ is generated in finite degrees and $Ext_A^n(S, S) = 0$, for all simple A-modules S and all $n >> 0$

Proof: We only need to prove 2) $\Longrightarrow$ 1). Let us fix some $s > 0$ such that $\oplus_{0 \leq i \leq s} Ext_A^i(A/J, A/J)$ generates E as a ring. We denote by r the number of nonisomorphic simple A-modules and fix $m > Sup\{k \geq 0 : Ext_A^k(T, T) \neq 0$, for some simple $_AT\}$, something which is possible to do since our hypothesis guarantees the existence of such a supremum. We claim that if $n > m \cdot r \cdot s$ then $Ext_A^n(A/J, A/J) = 0$ and, hence, assertion 1) will follow. Indeed, if $n > m \cdot r \cdot s$ then $Ext_A^n(A/J, A/J)$ is contained in a sum of products of the form $Ext_A^{i_1}(A/J, A/J) \cdot ... \cdot Ext_A^{i_t}(A/J, A/J)$, with $i_1 + ... + i_t = n$ and $i_k \leq s$ for $k = 1, ..., t$. From that we get that $n \leq t \cdot s$, which implies $t \geq \frac{n}{s} > \frac{m \cdot r \cdot s}{s} = m \cdot r$. But $Ext_A^{i_1}(A/J, A/J) \cdot ... \cdot Ext_A^{i_t}(A/J, A/J)$ is contained in a sum of products of the form $Ext_A^{i_1}(S_{t-1}, S_t) \cdot ... \cdot Ext_A^{i_t}(S_0, S_1)$, with the S_i simple. In any such product, there exists a simple S_i which appears repeated, at least, $m + 1$ times. Suppose $j_0 < j_1 < ... < j_m$ are different indices of $\{0, 1, ..., t\}$ such that $S_{j_0} = S_{j_1} = ... = S_{j_m}$, a simple A-module which we denote by X. In case the product $Ext_A^{i_1}(S_{t-1}, S_t) \cdot ... \cdot Ext_A^{i_t}(S_0, S_1)$ is nonzero, one gets that $Ext_A^j(X, X) \neq 0$, for some $j \geq m$, which is a contradiction. As a consequence, $Ext_A^n(A/J, A/J) = 0$ as desired.

Using now the remark, the following consequence is straightforward.

Corollary 1. *Let A be an Artin algebra with center K. The following assertions are equivalent:*

1. *The global dimension of A is finite*

2. *$E = Ext_A^*(A/J, A/J)$ is finitely generated as a K-algebra and $Ext_A^n(S, S) = 0$, for all simples A-modules S and all $n \gg 0$*

Example 1. *Let K be a field and A the monomial algebra with quiver $Q : 1 \rightleftarrows 2$ and the path $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 1 \xrightarrow{\alpha} 2$ as unique zero-relation. By [8], we know that $gl.dim(A) = \propto$ but $Ext_A^n(S, S) = 0$ for all simples S and all $n \gg 0$. According to the above theorem, the Yoneda algebra $E = Ext_A^*(A/J, A/J)$ cannot be finitely generated. That can be explicitly seen by using Anick and Green's resolution (cf. [2]). Using the terminology of this latter paper, the sets of chains are $\Gamma_n = Q_n$ $(n = 0, 1)$ and $\Gamma_n = \{\alpha(\beta\alpha)^{n-1}\}$, for all $n \geq 2$. By using [5][Theorem A, Proposition 1.2], we see that E can be represented by a quiver $\tilde{Q}$ having $\tilde{Q}_0 = Q_0$, one arrow $b : 1 \rightarrow 2$ corresponding to the 1-chain β and arrows $a_n : 2 \rightarrow 1$ $(n > 0)$, where a_n corresponds to the n-chain $\alpha(\beta\alpha)^{n-1}$. All possible products of arrows in $\tilde{Q}$ are zero and the grading on E assigns degree zero to the vertices, $deg(b) = 1$ and $deg(a_n) = n$, for all $n > 0$.*

We move now to study the case of Loewy length ≤ 3. The following lemma is probably known, but we include a proof for the sake of completeness.

Lemma 1. *Let A be a ring such that A/J is semisimple and $J^3 = 0$, where $J = J(A)$. Let $_AP$ be a projective module and M a submodule of JP. Then M decomposes as $M = N \oplus X \oplus Y$, where X is a semisimple direct summand of JP, Y is a (semisimple) direct summand of J^2P and N is a submodule containing no simple direct summand. Moreover, the following assertions are equivalent:*

1. $M \cap J^2P = JM$

2. $Y = 0$ *in the above decomposition*

Proof: Since $J^2M = 0$, JM is a semisimple submodule of M and we can fix a decomposition $Soc(M) = JM \oplus Z$. Then the canonical composition $Z \xrightarrow{j} M \xrightarrow{p} M/JM$ is a (necessarily split) mono. If now $\pi : M/JM \longrightarrow Z$ is a retraction for $p \circ j$, then $\pi \circ p$ is a retraction for j, so that, putting $N = Ker(\pi \circ p)$, we get a decomposition $M = N \oplus Z$. We claim that N contains no simple direct summand. To see that, notice first that $JM = JN$. If, by of contradiction, we assume that there exists a simple direct summand S of N, then we have a decomposition $N = S \oplus N'$, from which follows that $JM = JN = JN'$. Now we get a decomposition $Soc(M) = Soc(N') \oplus S \oplus Z$, with $JM \subseteq Soc(N')$. That contradicts the initial decomposition. Now, since Z is semisimple, we have a decomposition $Z = (Z \cap J^2P) \oplus X$. But $X \cap J^2P = 0$, which implies that the canonical composition $X \hookrightarrow JP \xrightarrow{pr} JP/J^2P$ is a (split) mono and, arguing as above, we conclude that X is a (semisimple) direct summand of JP. Next we prove the equivalence of assertions 1 and 2:

1) $\Longrightarrow$ 2) We always have inclusions $JM = JN \subseteq JN \oplus Y \subseteq M \cap J^2P$. Hence, assertion 1) implies that $Y = 0$

2) $\Longrightarrow$ 1) We just need to prove the inclusion $M \cap J^2P \subseteq JM$. Take $m \in M \cap J^2P$, which we decompose as $m = n + x$, with $n \in N$ and $x \in X$. Since $Jm = 0 = Jx$ we get that $Jn = 0$ and An is a semisimple submodule of N. We claim that $n \in JM$. Indeed, if $n \notin JM = JN$, then the semisimplicity of An yields a decomposition $An = (An \cap JN) \oplus Aan$, with $an \neq 0$. Now the canonical composition $Aan \hookrightarrow N \xrightarrow{p} N/JN$ is monic and, arguing as above, we get that Aan is a nonzero (semisimple) direct summand of N, which is a contradiction, thus settling our claim. Since $n \in JM \subseteq M \cap J^2P$, we get $x \in J^2P$. But, since Ax is a semisimple direct summand of JP, the latter implies that $x = 0$ and, hence, $m = n \in JM$.

Using the above lemma, to each simple A-module S we can inductively associate a sequence of decompositions $\Omega^n S = M_n \oplus Z_n$ as follows. We put $M_n = \Omega^n S$, $Z_n = 0$ for $n = 0,1$. If $n > 1$ and the decomposition $\Omega^{n-1}S = M_{n-1} \oplus Z_{n-1}$ is already defined, then we put $P'_{n-1} = P(M_{n-1})$ and $P''_{n-1} = P(Z_{n-1})$. According to the above lemma, we have a decomposition $\Omega M_{n-1} = M_n \oplus Y_n$, where Y_n is a direct summand of $J^2P'_{n-1}$ and $M_n \cap J^2P'_{n-1} = JM_n$. Then, by

putting $Z_n = Y_n \oplus \Omega Z_{n-1}$, we get the desired decomposition $\Omega^n S = M_n \oplus Z_n$. In this way we get a sequence of modules $(S = M_0, M_1, M_2, ...)$ which is uniquely determined by S. When we look at the canonical isomorphism $Ext_A^n(S, A/J) \cong Hom_A(\Omega^n S, A/J)$ as an identification, we can view $Hom_A(M_n, A/J)$ as a subgroup of $Ext_A^n(S, A/J)$. That is the sense of the following crucial result.

Lemma 2. *Let A be a ring such that A/J is semisimple and $J^3 = 0$. With the above notation, $Hom_A(M_n, A/J)$ is contained in $Ext_A^1(A/J, A/J) \cdot .^n. \cdot Ext_A^1(A/J, A/J) \cdot Ext_A^0(S, A/J)$*

Proof: Our argument is inspired by that of [4][Proposition 3.2]. We apply induction on n, the case $n = 0$ being trivially true. Suppose $n > 0$. We plan to prove that $Hom_A(M_n, A/J) \subseteq Ext_A^1(A/J, A/J) \cdot Hom_A(M_{n-1}, A/J)$ (after the above mentioned identifications) and the induction hypothesis will give the desired result. Let $f \in Hom_A(M_n, A/J)$. Then $(f\ 0) : M_n \oplus Z_n = \Omega^n S \longrightarrow A/J$ represents an element of $Ext_A^n(S, A/J)$. The canonical inclusion $\iota : \Omega^n S = M_n \oplus Z_n \hookrightarrow JP'_{n-1} \oplus JP''_{n-1}$ can be written as a matrix $\iota =$

$$\begin{pmatrix} i_{11} & i_{12} \\ 0 & i_{22} \end{pmatrix}$$

where $i_{11} : M_n \hookrightarrow JP'_{n-1}$ is the inclusion, i_{22} is the map $(0\ j) : Z_n = Y_n \oplus \Omega Z_{n-1} \longrightarrow JP''_{n-1}$, with $j : \Omega Z_{n-1} \longrightarrow JP''_{n-1}$ the canonical inclusion, and i_{12} is the map $(i\ 0) : Z_n = Y_n \oplus \Omega Z_{n-1} \longrightarrow JP'_{n-1}$, with $i : Y_n \hookrightarrow JP'_{n-1}$ the canonical inclusion. Notice that, by definition, we have $Im(i_{12}) \subseteq J^2 P'_{n-1}$. As a consequence, the induced homomorphism $\Omega^n S/J\Omega^n S = (M_n/JM_n) \oplus (Z_n/JZ_n) \longrightarrow (JP'_{n-1}/J^2 P'_{n-1}) \oplus (JP''_{n-1}/J^2 P''_{n-1})$ has a diagonal matrix shape

$$\begin{pmatrix} \bar{i}_{11} & 0 \\ 0 & \bar{i}_{22} \end{pmatrix}$$
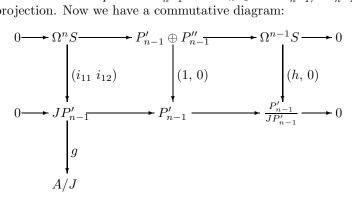
.

On the other hand, f factors as $f = \bar{f} \circ p_M$, where $p_M : M_n \longrightarrow M_n/JM_n$ is the canonical projection, and $\bar{i}_{11} : M_n/JM_n \longrightarrow JP'_{n-1}/J^2 P'_{n-1}$ is a (split) monomorphism by the definition of M_n. If we choose a retraction α for $\bar{i}_{11}$ and put $\bar{g} = \bar{f} \circ \alpha$, then the induced morphism $(\bar{g}\ 0) : (JP'_{n-1}/J^2 P'_{n-1}) \oplus (JP''_{n-1}/J^2 P''_{n-1}) \longrightarrow A/J$ has the property that $(\bar{g}\ 0)\circ$

$$\begin{pmatrix} \bar{i}_{11} & 0 \\ 0 & \bar{i}_{22} \end{pmatrix}$$

$= (\bar{f}\ 0)$.

We now denote by g the composition $JP'_{n-1} \xrightarrow{pr} JP'_{n-1}/J^2 P'_{n-1} \xrightarrow{\bar{g}} A/J$ and then the morphism $(g\ 0) : JP'_{n-1} \oplus JP''_{n-1} \longrightarrow A/J$ has the property that $(g\ 0)\circ\iota = (f\ 0)$. In particular, $g\circ i_{11} = f$ and $g\circ i_{12} = 0$. By consicering $(i_{11}\ i_{12})$ with arrival in P'_{n-1} instead of JP'_{n-1}, we have $M_{n-1} \cong Coker[(i_{11}\ i_{12})]$ and, since the composition $\pi \circ (i_{11}\ i_{12})$ is zero, where $\pi : P'_{n-1} \longrightarrow P'_{n-1}/JP'_{n-1}$ is the canonical projection, we have a unique morphism $h : M_{n-1} \longrightarrow P'_{n-1}/JP'_{n-1}$

such that the composition $P'_{n-1} \xrightarrow{pr} M_{n-1} \xrightarrow{h} P'_{n-1}/JP'_{n-1}$ is the canonical projection. Now we have a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^n S & \longrightarrow & P'_{n-1} \oplus P''_{n-1} & \longrightarrow & \Omega^{n-1}S & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle (i_{11}\ i_{12})} & & \downarrow{\scriptstyle (1,\,0)} & & \downarrow{\scriptstyle (h,\,0)} & & \\
0 & \longrightarrow & JP'_{n-1} & \longrightarrow & P'_{n-1} & \longrightarrow & \frac{P'_{n-1}}{JP'_{n-1}} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle g} & & & & & & \\
& & A/J & & & & & &
\end{array}
$$

where the rows are the obvious exact sequences and the composition of the vertical left arrows is $(f,\,0)$. Now, since $JP'_{n-1} = \Omega(P'_{n-1}/JP'_{n-1})$, we have $g \in Ext^1_A(P'_{n-1}/JP'_{n-1}, A/J)$ and, on the other hand, we also have $(h\ 0) \in Ext^{n-1}_A(S, P'_{n-1}/JP'_{n-1})$. Then the Yoneda product $g \cdot (h\ 0)$ is just $g \circ (i_{11}\ i_{12}) = (f\ 0)$. Moreover, $(h\ 0)$ clearly belongs to $Hom_A(M_{n-1}, P'_{n-1}/JP'_{n-1})$ when we view the latter as a subgroup of $Ext^{n-1}_A(S, P'_{n-1}/JP'_{n-1})$. From that we get the desired inclusion $Hom_A(M_n, A/J) \subseteq Ext^1_A(A/J, A/J) \cdot Hom_A(M_{n-1}, A/J)$.

We are now in a position to prove:

Theorem 2. *Let A be a ring such that A/J is semisimple and $J^3 = 0$. The following assertions are equivalent:*

1. *The global dimension of A is finite*

2. *$Ext^n_A(S, S) = 0$, for all simple A-modules S and all $n \gg 0$*

3. *Every simple A-module has either finite projective dimension or finite injective dimension*

Proof: We only need to prove 2) $\Longrightarrow$ 1). If condition 2) holds, even without the hypothesis that $J^3 = 0$, one can define an order relation among the simple modules by the rule: $S \preceq T$ iff S is a direct summand of $\Omega^m T$, for some $m \geq 0$. Indeed, that is always a preorder relation and we only need to check antisymmetry. If $S \preceq T \preceq S$ then S is a direct summand of $\Omega^m T$ and T is a direct summand of $\Omega^n S$, for some $m, n \in \mathbf{N}$. From that it follows that S is a direct summand of $\Omega^{m+n}S$, and hence a direct summand of $\Omega^{(m+n)k}S$ for all $k \in \mathbf{N}$. If $m + n > 0$ we get a contradiction with the fact that $Ext^r_A(S, S) = 0$ for all $r \gg 0$. Therefore $m + n = 0$ or, equivalently, $m = n = 0$, which means that $S = T$ as desired.

We next claim that, for every simple module S, the sequence $(S = M_0, M_1, M_2, ...)$ defined immediately before the previous lemma is finite, i.e., there is a $t > 0$ such that $M_t = 0$ (and, hence, $M_n = 0$ for all $n \geq t$). To see that, we follow

an argument analogous to that of Theorem 1. Let r be the number of nonisomorphic simple A-modules and fix $m > Sup\{k \geq 0 : Ext_A^k(T,T) \neq 0$, for some simple $_A T\}$. Now we claim that if $n > m \cdot r$, then $M_n = 0$. Indeed, if $M_n \neq 0$ then $Hom_A(M_n, A/J) \neq 0$ and the foregoing lemma implies that $Ext_A^1(A/J, A/J) \cdot .^n_. \cdot Ext_A^1(A/J, A/J) \neq 0$, from which we deduce the existence of simple A-modules $S_0, S_1, ..., S_n$ such that $Ext_A^1(S_{n-1}, S_n) \cdot ... \cdot Ext_A^1(S_0, S_1) \neq 0$. But the sequence of simples $(S_0, S_1, ..., S_n)$ will necessarily have one simple repeated, at least, $m + 1$ times and, as in the proof of Theorem 1, we get a simple X such that $Ext_A^j(X, X) \neq 0$ for some $j \geq m$, which is a contradiction. Observe that, when S is minimal with respect to the order relation $\preceq$, one has $\Omega^n S = M_n$ for all $n \geq 0$. Hence, our argument also proves that $pd_A(S) < \propto$ in that case.

Let now S be an arbitrary simple module and let us recall that, by definition of the associated sequence $(S = M_0, M_1, M_2, ...)$, we have $\Omega M_{n-1} = M_n \oplus Y_n$, where Y_n is a direct summand $J^2 P(M_{n-1})$, for all $n > 0$. From that we get, for every fixed $n > 0$, that $pd_A(S) < \propto$ if, and only if, the dimensions $pd(M_n)$, $pd_A(Y_1),...,pd_A(Y_n)$ are all finite. Bearing in mind the finiteness of the sequence (M_n), we conclude that $pd_A(S) \propto$ iff $pd_A(Y_n) \propto$, for all $n > 0$. But each Y_n is a direct sum of simple modules which are strict predecessors of S with respect to the order relation $\preceq$. Since the minimal elements with respect to $\preceq$ have finite projective dimension, an easy induction procedure shows that $pd_A(S) < \propto$, for every simple left A-module and, hence, $gl.dim(A) < \propto$.

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